mapping. ... conformal

P.G.-sem 2nd paper-VI, unit IV Theorems on conformal mapping.

Theorem: Necessary Condition for $w = f(z)$ to represent a conformal mapping (1)

If the mapping $w = f(z)$ is conformal, then the function $f(z)$ is an analytic function of z.

Proof. Suppose that $w = u(x,y) + iv(x,y) = f(z)$

Also suppose that $u = u(x,y)$

$v = v(x,y)$

are equations which define conformal transformation from z-plane to w-plane.

Let ds and $d\sigma$ be the line element in z-plane and w-plane respectively so that

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	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

$$ds^2 = dx^2 + dy^2$$
$$d\sigma^2 = du^2 + dv^2$$
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

~~So that~~ – Squaring and adding

$$= \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]dx^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dy^2$$
$$+ 2\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]dx\,dy$$

Since the transformation is conformal and hence the ratio $\frac{d\sigma^2}{ds^2}$ is independent of direction. Comparing the last equation with $ds^2 = dx^2 + dy^2$

$$\frac{u_x^2 + v_x^2}{1} = \frac{u_y^2 + v_y^2}{1} = \frac{u_x u_y + v_x v_y}{0}$$

$$\Rightarrow \begin{cases} u_x^2 + v_x^2 = u_y^2 + v_y^2 \quad \text{———} (1) \\ u_x u_y + v_x v_y = 0 \quad \text{———} (2) \end{cases}$$

From (2) $\frac{u_x}{v_y} = \frac{v_x}{-u_y} = \lambda$, say

$$\Rightarrow \left.\begin{array}{l} u_x = \lambda v_y \\ \text{and } v_x = -\lambda u_y \end{array}\right\} \text{———} (3)$$

Putting in (1) [redacted] $\lambda^2 v_y^2 + \lambda^2 u_y^2 = u_y^2 + v_y^2$

$$\Rightarrow (\lambda^2 - 1)\left[u_y^2 + v_y^2\right] = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

$\therefore$ From (3) $u_x = v_y,\ u_y = -v_x$ (when $\lambda = 1$) (4)

$u_x = -v_y,\ u_y = v_x$ (when $\lambda = -1$) (5)

The equations (4) are Cauchy-Riemann equations and hence $w = f(z)$ is an analytic function. The equation (5) are reduced to (4) when we write $-v$ in place of v i.e, by taking as image figure obtained by the reflection in the real axis of w-plane.

In other words, we have to take as image figure found by reflection in the real axis of the w-plane, (i.e. in this case the magnitude of the angle will be preserved but the direction will be changed). Hence the equations (5) correspond to an isogonal transformation.

Therefore, it is established that if the transformation
$$w = f(z)$$
is conformal, then $f(z)$ must be an analytic function of z. //

Q. (Theorem 3) State and prove the n. & s. c. for $w = f(z)$ to represent a conformal transformation

(State & prove theorem & theorem 2)